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The interference of two passive scalars in a homogeneous isotropic turbulent field

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In this work, we study the interference of two scalars diffusing in homogeneous isotropic turbulence. We use the method described by Kaplan & Dinar (1988*b*) to calculate the cross-correlation coefficient ρ between the concentration fluctuations of two sources. The dependence of ρ on the source separation, shapes and sizes, and its time evolution is calculated for different points in space. Results for the case of two line sources are compared with data from wind tunnel experiments (Warhaft 1984), and seen to be in good agreement. At the centreline ρ is shown to increase as overlap of the two plumes increases. ρ may be either negative or positive depending on the separation between the two sources.

1. Introduction

The cross-correlation of two or more scalars, diffusing in a turbulent medium, is of great interest from both theoretical and practical points of view. Many physical quantities in the atmosphere or in the ocean depend on the fluctuations of two or more scalars and their temporal and spatial distribution can be determined only if the covariance of these two quantities is known. Examples of such quantities are: the radio refractive index which depends on the temperature–humidity covariance, the temperature–salinity covariance in the ocean and the mixing of two reactants which determines the rate of reaction.

The theoretical approach to the problem of diffusion in turbulent media is usually limited to the prediction of scalar averages and is not normally used to predict higher moments of the scalar fluctuations. (See for example Taylor 1921; Hanna 1979; Thomson 1986.) Only in recent years with the development of the theory of two-particle statistics, has the second moment of the fluctuations become more readily calculable (Durbin 1980; Sawford 1983; Sawford & Hunt 1986; Kaplan & Dinar 1988*a*, *b*). Those approaches, when formulated in the one-dimensional case are not consistent with the incompressibility constraint and the predicted fluctuation distribution is affected by the fluid density fluctuations. In order to overcome this limitation, the model suggested by Kaplan & Dinar (1988*a*) was extended to the three-dimensional case and constrained to fulfil the continuity equation. In the present work, we use this three-dimensional model to predict the spatial and temporal evolution of the correlation of two scalar fluctuations. The prediction of the model is compared with the wind tunnel measurements of Warhaft (1984).

The outline of the paper is as follows. After a brief description of the three-dimensional model, we calculate the cross-correlation ρ as a function of space and time for various sources (shapes and magnitudes), and for different values of their separation. Finally, results for the two line sources are compared to the wind tunnel experiments of Warhaft (1984), and to field experiments of Sawford, Frost & Allan (1985).

2. Description of the model

2.1. *Basic equations*

The three-dimensional model suggested by Kaplan & Dinar (1988*b*) to describe the two-particle statistics in homogeneous isotropic turbulence is based on the Lagrangian statistics approach. It is assumed that the motion of the two particles in the field is described by the following equations:

$$\frac{\mathrm{d}\boldsymbol{r}_i}{\mathrm{d}t} = \boldsymbol{v}_i(t) \quad (i = 1,2), \tag{2.1}$$

$$\boldsymbol{v}_i(t+\Delta t) = R_{\mathrm{L}}(\Delta t)\,\boldsymbol{v}_i(t) + (1-R_{\mathrm{L}}^2(\Delta t))^{\frac{1}{2}}\boldsymbol{\theta}(\boldsymbol{r}_i(t)), \tag{2.2}$$

where $R_{\mathrm{L}}(\Delta t)$ is the Lagrangian time correlation function given by

$$R_{\mathrm{L}}(\Delta t) = \exp(-\Delta t/T_{\mathrm{L}}). \tag{2.3}$$

T_{L} is the Lagrangian timescale. $\boldsymbol{\theta}(\boldsymbol{r}_i)$ is a spatially correlated white noise. We assume that the spatial covariance of the random field $\boldsymbol{\theta}(\boldsymbol{r})$ is given by the Eulerian covariance function of the turbulence field, $C(\boldsymbol{r}_i-\boldsymbol{r}_j)$, i.e.

$$\langle\theta_\alpha(\boldsymbol{r}_i)\,\theta_\beta(\boldsymbol{r}_j)\rangle = C_{\alpha\beta}(\boldsymbol{r}_i-\boldsymbol{r}_j) \quad (\alpha,\beta = X,Y,Z). \tag{2.4}$$

In homogeneous isotropic turbulence the covariance function is determined by its projection in a given direction, lets say the X-direction (see Kaplan & Dinar 1988*b*). Given the one-dimension covariance function $C_{XX}(X,0,0) = \sigma_v^2 f(X)$, the covariance matrix $C_{\alpha\beta}$ can be determined:

$$C_{\alpha\beta} = \sigma_v^2[-0.5\frac{r_\alpha r_\beta}{r}f'(r) + \delta_{\alpha\beta}(f(r)+0.5rf'(r))] \tag{2.5}$$

The one-dimensional function $f(X)$ is characterized by the Eulerian lengthscale L_{E} and by the Kolomgorov spectra for small X in the inertial subrange. Following Durbin (1980), we chose $f(X)$ to be

$$f(X) = 1-\left(\frac{X^2}{X^2+L_{\mathrm{E}}^2}\right)^{\frac{1}{3}}. \tag{2.6}$$

The integral Eulerian lengthscale of this function is approximately $0.747L_{\mathrm{E}}$. The covariance matrix $C_{\alpha\beta}$ described by (2.5) is compatible with incompressibility and isotropy of the turbulent medium (see the Appendix). A method of constructing such a random field $\boldsymbol{\theta}(\boldsymbol{r})$ with covariance given by (2.5) is described in Kaplan & Dinar (1988*b*). Given the covariance matrix $C(\boldsymbol{r}_i-\boldsymbol{r}_j)$ and initial conditions, (2.1) and (2.2) can be solved and trajectories of the two particles can be calculated.

2.2. *Definition of the concentration fluctuations at a given point*

We adopted the approach of concentration fluctuations at high Péclet number suggested by Durbin (1980, 1982). According to this definition, the concentration at a given point $\boldsymbol{r}$ at time t is given by averaging the instantaneous concentration over a small volume V_η of order η around $\boldsymbol{r}$, i.e.

$$C(\boldsymbol{r},t) = \frac{1}{V_\eta}\iiint_{V_\eta}\tilde{C}(\boldsymbol{r}',t)\,\mathrm{d}^3\boldsymbol{r}', \tag{2.7}$$

where $\tilde{C}(\boldsymbol{r},t)$ is the instantaneous point concentration and η is the Kolmogorov lengthscale. Using this definition, we take into account smearing by molecular action. The concentration fluctuations are determined primarily by the dynamics of large eddies and eddies in the inertial sub-range.

2.3. *Calculation of the moments of fluctuations*

It was proved by Egbert & Baker (1984) that in an incompressible flow, the joint probability $P_N(\boldsymbol{r}_1^0 \ldots \boldsymbol{r}_N^0; 0; \boldsymbol{r}_1 \ldots \boldsymbol{r}_N, t)$ for N particles located at $\boldsymbol{r}_1^0 \ldots \boldsymbol{r}_N^0$ at time $t = 0$ to be at locations $\boldsymbol{r}_1 \ldots \boldsymbol{r}_N$ respectively at time t (forward diffusion), is equal to the joint probability $P_N(\boldsymbol{r}_1 \ldots \boldsymbol{r}_N; t; \boldsymbol{r}_1^0 \ldots \boldsymbol{r}_N^0; 0;)$ that N particles which are at $\boldsymbol{r}_1 \ldots \boldsymbol{r}_N$ at time t came from locations $\boldsymbol{r}_1^0 \ldots \boldsymbol{r}_N^0$ respectively at time $t = 0$ (reversed diffusion). Using this theorem, the moments of concentration distribution can be calculated in terms of the reversed diffusion (see Egbert & Baker 1984). The Nth moment is given by

$$\begin{aligned} M^{(N)} &= \iiint P_N(\boldsymbol{r}_1^0, \boldsymbol{r}_2^0, \ldots \boldsymbol{r}_N^0, 0; \boldsymbol{r}, \boldsymbol{r}, \ldots \boldsymbol{r}, t)\, S(\boldsymbol{r}_1^0) S(\boldsymbol{r}_2^0) \ldots S(\boldsymbol{r}_N^0)\, \mathrm{d}^3\boldsymbol{r}_1^0 \ldots \mathrm{d}^3\boldsymbol{r}_N^0 \\ &= \iiint P_N(\boldsymbol{r}, \boldsymbol{r}, \ldots \boldsymbol{r}, t; \boldsymbol{r}_1^0, \boldsymbol{r}_2^0, \ldots \boldsymbol{r}_N^0, 0)\, S(\boldsymbol{r}_1^0) S(\boldsymbol{r}_2^0) \ldots S(\boldsymbol{r}_N^0)\, \mathrm{d}^3\boldsymbol{r}_1^0 \ldots \mathrm{d}^3\boldsymbol{r}_N^0, \end{aligned} \tag{2.8}$$

where $S(\boldsymbol{r})$ is the source distribution function. Denote by S_A a source distribution function centred at $\boldsymbol{r}_A$ and by S_B a source distribution function centred at $\boldsymbol{r}_B$. The second moment of concentration resulting from those two different sources is given by

$$\begin{aligned} C_{AB} &= \iint P_2(\boldsymbol{r}_1^0, \boldsymbol{r}_2^0, 0; \boldsymbol{r}, \boldsymbol{r}, t)\, S_A(\boldsymbol{r}_1^0)\, S_B(\boldsymbol{r}_2^0)\, \mathrm{d}^3\boldsymbol{r}_1^0\, \mathrm{d}^3\boldsymbol{r}_2^0 \\ &= \iint P_2(\boldsymbol{r}, \boldsymbol{r}, t; \boldsymbol{r}_1^0, \boldsymbol{r}_2^0, 0)\, S_A(\boldsymbol{r}_1^0)\, S_B(\boldsymbol{r}_2^0)\, \mathrm{d}^3\boldsymbol{r}_1^0\, \mathrm{d}^3\boldsymbol{r}_2^0. \end{aligned} \tag{2.9}$$

The integrals (2.8) and (2.9) are calculated using a Monte-Carlo method. In order to calculate $P(\boldsymbol{r}_1, \boldsymbol{r}_2; t; \boldsymbol{r}_1^0, \boldsymbol{r}_2^0; 0)$ one should follow the trajectories of particles backward in time. In stationary turbulence, this is equivalent to solving (2.1) and (2.2) forward in time (see Durbin 1980).

We start with a particle pair near the point $\boldsymbol{r}$ separated by a very small distance $\Delta\boldsymbol{r}$. Solving (2.1) and (2.2), we calculate their locations $\boldsymbol{r}_1^0$, $\boldsymbol{r}_2^0$ respectively, after time t. Then we assign to particle 1, the source concentration $S_A(\boldsymbol{r}_1^0)$, and to particle 2 the concentration $S_B(\boldsymbol{r}_2^0)$. Repeating this procedure M times, and assuming $S_A(\boldsymbol{r}) = S(\boldsymbol{r}-\boldsymbol{r}_A)$, $S_B(\boldsymbol{r}) = S(\boldsymbol{r}-\boldsymbol{r}_B)$, the integrals for the moments can be approximated by

$$\left.\begin{aligned} M_A^{(1)} &= \frac{1}{2M} \sum_j \left(S_A(\boldsymbol{r}_1^{0(j)}) + S_A(\boldsymbol{r}_2^{0(j)})\right), \\ M_B^{(1)} &= \frac{1}{2M} \sum_j \left(S_B(\boldsymbol{r}_1^{0(j)}) + S_B(\boldsymbol{r}_2^{0(j)})\right). \end{aligned}\right\} \tag{2.10}$$

$M_A^{(1)}$, $M_B^{(1)}$ are the averaged concentrations at point $\boldsymbol{r}$ contributed by the sources located at $\boldsymbol{r}_A$, $\boldsymbol{r}_B$, respectively. And

$$\left.\begin{aligned} M_A^{(2)} &= \frac{1}{2M} \sum_j S_A(\boldsymbol{r}_1^{0(j)})\, S_A(\boldsymbol{r}_2^{0(j)}), \\ M_B^{(2)} &= \frac{1}{2M} \sum_j S_B(\boldsymbol{r}_1^{0(j)})\, S_B(\boldsymbol{r}_2^{0(j)}), \end{aligned}\right\} \tag{2.11}$$

where $M_A^{(2)}$, $M_B^{(2)}$ are the second moments of the concentration fluctuations at point $\boldsymbol{r}$, contributed by the sources located at $\boldsymbol{r}_A$, $\boldsymbol{r}_B$, respectively.

C_{AB} is the second moment of the concentration contributed by the sources located at $\boldsymbol{r}_A$ and at $\boldsymbol{r}_B$, calculated at point $\boldsymbol{r}$.

$$C_{AB} = \frac{1}{2M}\sum_j [S_A(\boldsymbol{r}_1^{0(j)})\,S_B(\boldsymbol{r}_2^{0(j)}) + S_A(\boldsymbol{r}_2^{0(j)})\,S_B(\boldsymbol{r}_1^{0(j)})]. \tag{2.12}$$

The variances of concentration fluctuations contributed by source A and B are V_A, V_B respectively:

$$V_A = M_A^{(2)} - [M_A^{(1)}]^2, \qquad V_B = M_B^{(2)} - [M_B^{(1)}]^2. \tag{2.13}$$

With these moments, the correlation between concentration from sources A and B at point $\boldsymbol{r}$ is calculated:

$$\rho = \frac{C_{AB} - M_A^{(1)} M_A^{(2)}}{(V_A V_B)^{\frac{1}{2}}}. \tag{2.14}$$

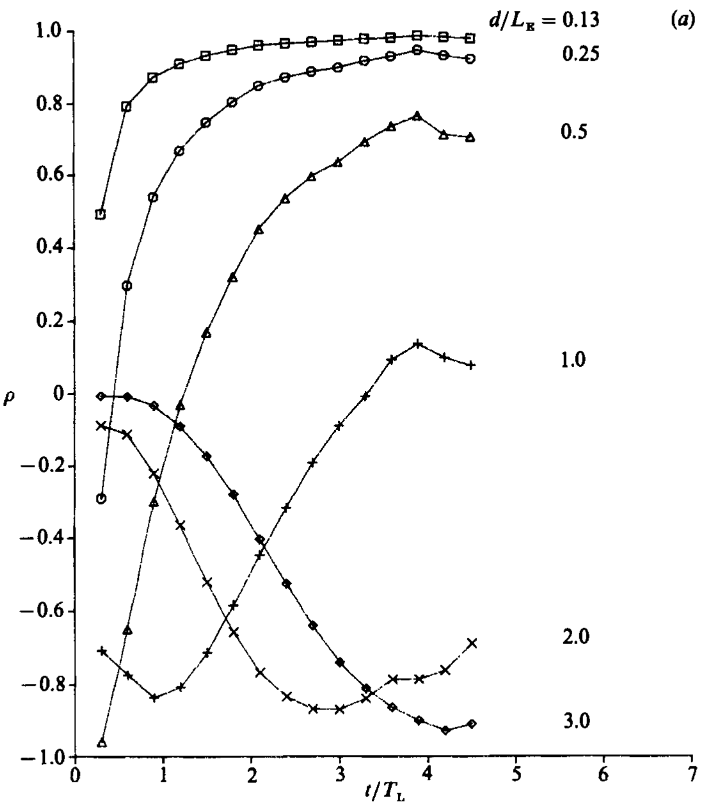
The fluctuation intensities s_A, s_B are

$$s_A = \frac{V_A^{\frac{1}{2}}}{M_A^{(1)}}, \qquad s_B = \frac{V_B^{\frac{1}{2}}}{M_B^{(1)}}. \tag{2.15}$$

3. Results and discussion

Figure 1 shows the evolution in time of ρ at the centreline between the two sources (plane source, line source and sphere source) for various values of d, the spacing of the sources. The source shape used is of a Gaussian distribution with $\sigma = 0.1L_{\mathrm{E}}$. The time t is scaled by the Lagrangian timescale T_{L}, and d is scaled by the Eulerian lengthscale L_{E}. Figure 2 shows the dependence of ρ on the spacing between the sources, for different values of t. The values of ρ at large times are less accurate owing to statistical noise and the wiggles that appear in the curves are result of this.

On the basis of physical intuition, one expects the time evolution of the correlation between two sources to be as follows: at small times, when the cloud width is very small compared to the distance between the two sources, neither cloud is present for most of the time at the detector, then ρ is small. Later in time, when the cloud grows and its width is of the order of magnitude of the space between the two sources, ρ becomes negative. This can be understood by looking at figure 3. In those realizations where the cloud moves toward source A, the concentration contributed by source A at the centreline is small while that contributed by source B increases, and vice versa where the cloud moves toward B. When the time is large enough so that the cloud widths are larger than the spacing between the two sources, the two clouds are well mixed and the correlation between their concentrations tends to 1. This description suggests that the time evolution of the correlation can be scaled and is a unique function of the ratio $\sigma(t/T_{\mathrm{L}})/d$, where $\sigma(t/T_{\mathrm{L}})$ is given in (4.2) and d is the space between the two sources. This scaling is presented in figure 4 for the three sources. Results indicate that ρ is a unique function of $\sigma(t/T_{\mathrm{L}})/d$. The behaviour of the correlation as function of time, as described above, is similar for the three sources (figure 4): spherical source, line source and plane source. The minimum is lowest in the case of a plane source, where the motion of the plume has only one degree of freedom. In figure 5 (solid line), we present the dependence of the correlation on the cross-wind direction. It can be seen that the correlation increases with the distance from the centreline. This behaviour can be understood by looking at figure 3. If we look at a line far from the centreline (for example, the dotted line E in the figure), we find that at this point, the two concentrations increase or decrease together as the plumes move.



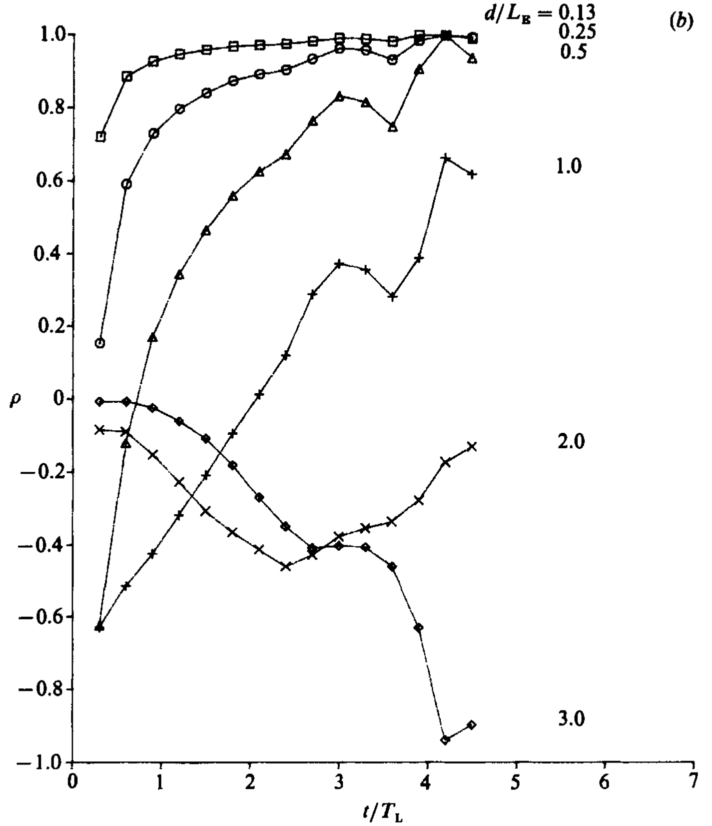


FIGURE 1 (*a*, *b*). For caption see next page.

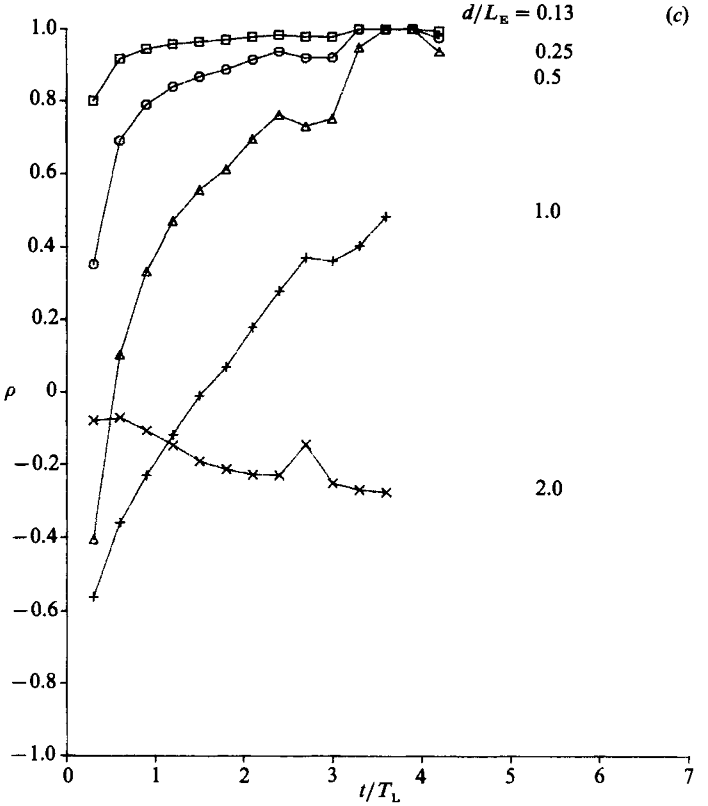


FIGURE 1. Evolution in time of ρ at the centreline between two sources in a turbulent flow for various values of the spacing between the sources. (*a*) Plane sources; (*b*) line sources; (*c*) sphere sources.

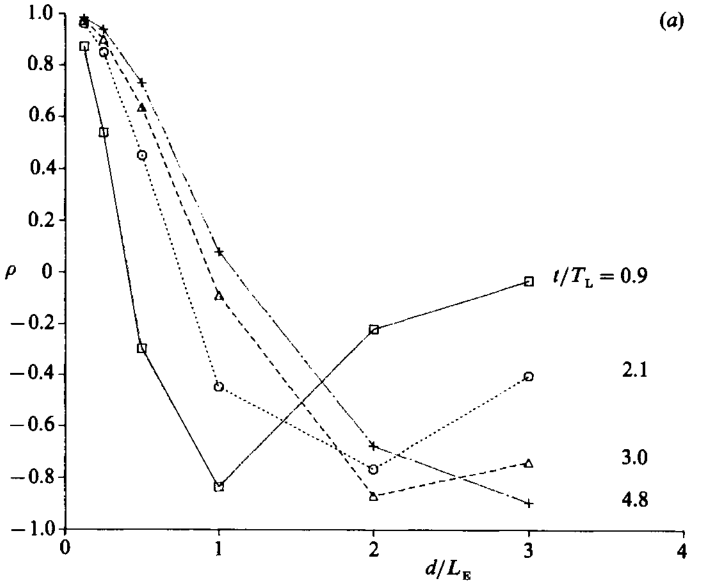


FIGURE 2(*a*). For caption see facing page.

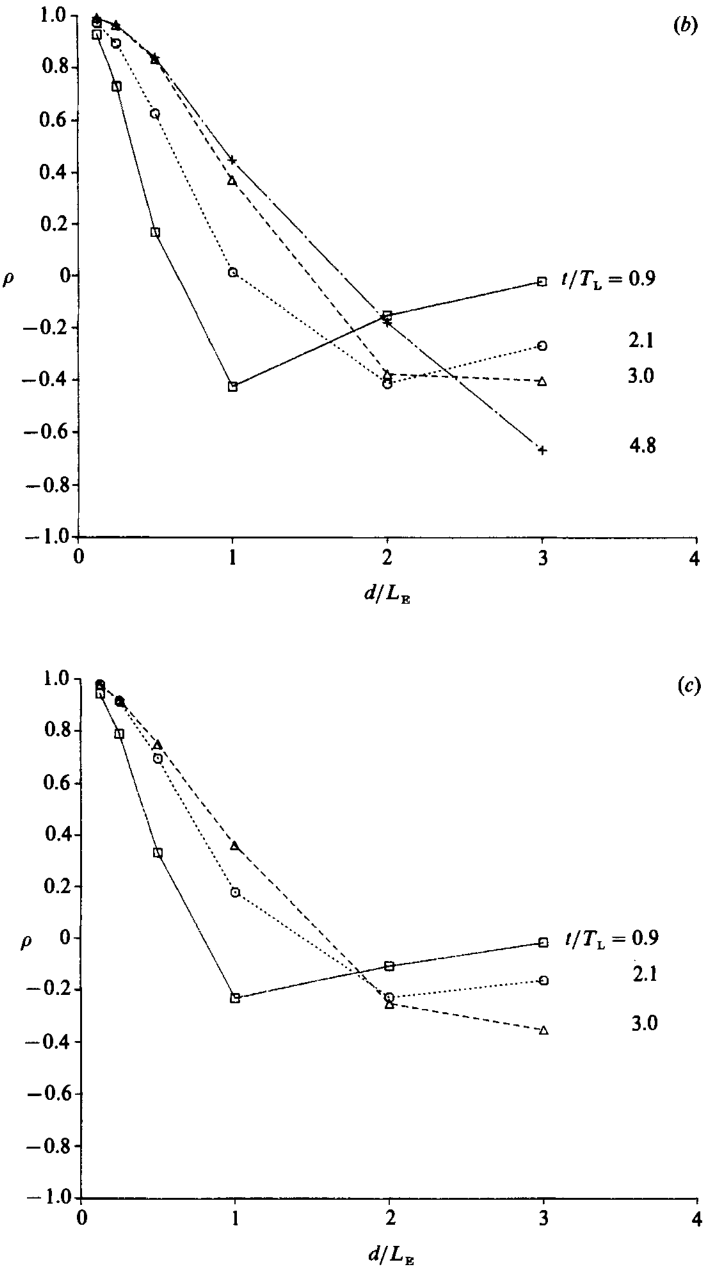


FIGURE 2. Dependence of ρ on the spacing between the sources for different time values. (*a*) Plane sources; (*b*) line sources; (*c*) sphere sources.

4. Comparison with experiments

Experiments that give information on ρ and its dependence on various parameters have been performed recently (Sawford *et al.* 1985; Warhaft 1984). The more detailed information is given in the experiment of Warhaft (1984), which studies the interference of passive thermal fields produced by two line sources in a decaying grid turbulence. The evolution of ρ in time and its dependence on the source spacing, as

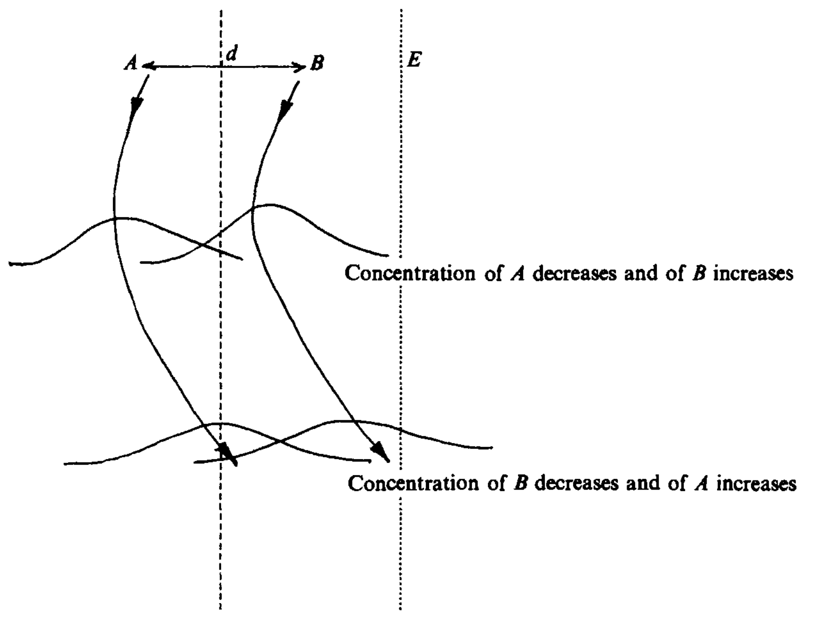


FIGURE 3. Description of the behaviour of plumes from two separated sources.

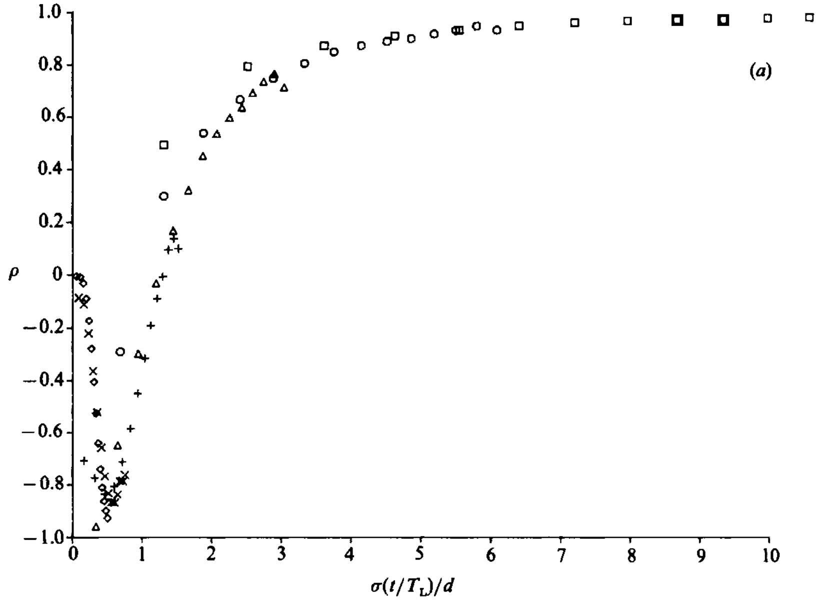


FIGURE 4 (a). For caption see facing page.

observed in this experiment, is qualitatively the same as described by our model. The difficulty in comparing the model to the experiment quantitatively is because the theoretical model describes stationary turbulence while the measurements are taken in a decaying turbulent field. In order to compare Warhaft's results to our model, we define an equivalent Lagrangian timescale T_L as the averaged Lagrangian timescale

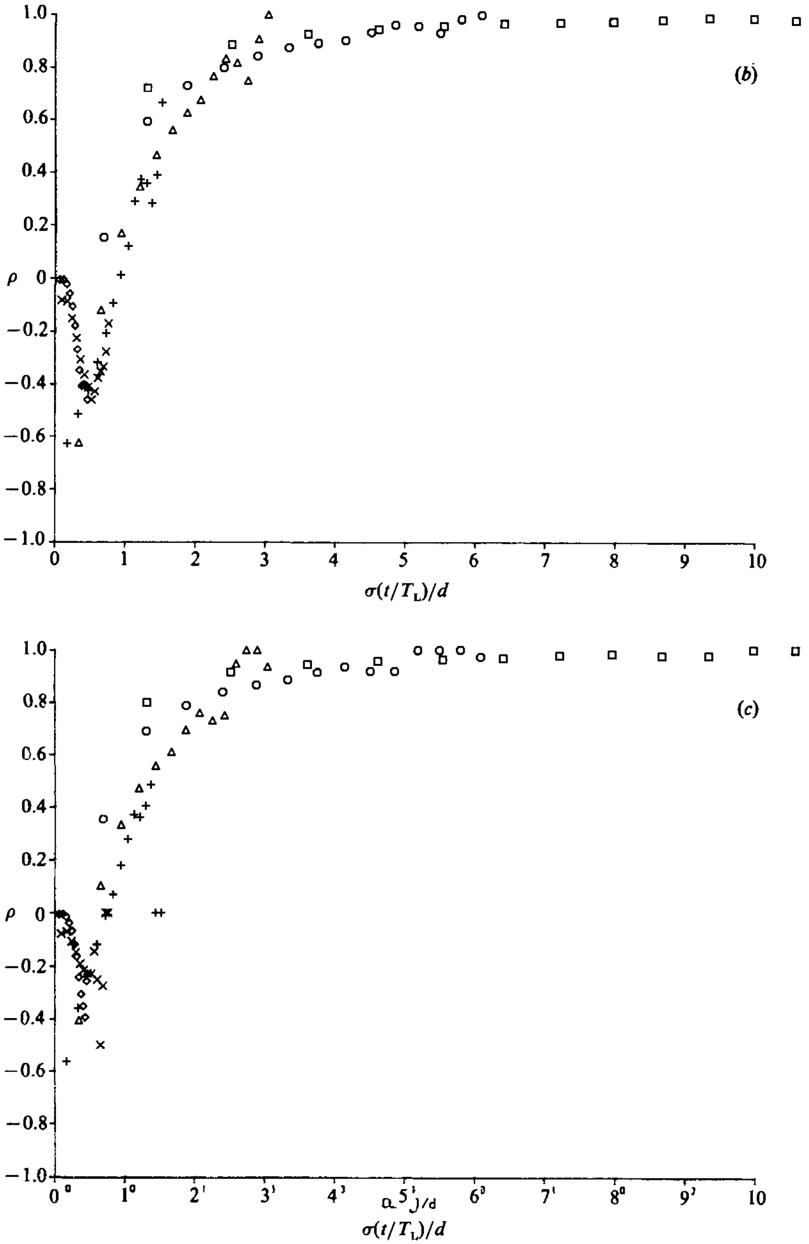


FIGURE 4. Evolution in time of ρ as a function of $\sigma(t/T_{\mathrm{L}})/d$. (*a*) Plane source; (*b*) line source; (*c*) sphere source.

over the life time of the cloud. The relation between T_{L}, σ_v and the integral Eulerian lengthscale L_{E} are assumed to be $T_{\mathrm{L}}\sigma_v/L_{\mathrm{E}} = 0.6$ (see Hanna 1981).

$$\tilde{T}_{\mathrm{L}} = \frac{1}{X_0 - X}\int_{X_0}^{X} 0.6\frac{L_{\mathrm{E}}(\xi)}{\sigma_v(\xi)}\mathrm{d}\xi, \tag{4.1}$$

where X_0 is the source location, X is the downwind distance from the grid, $L_{\mathrm{E}}(\xi)$ is the turbulence scale as a function of distance from the grid and $\sigma_v(\xi)$ is the turbulence

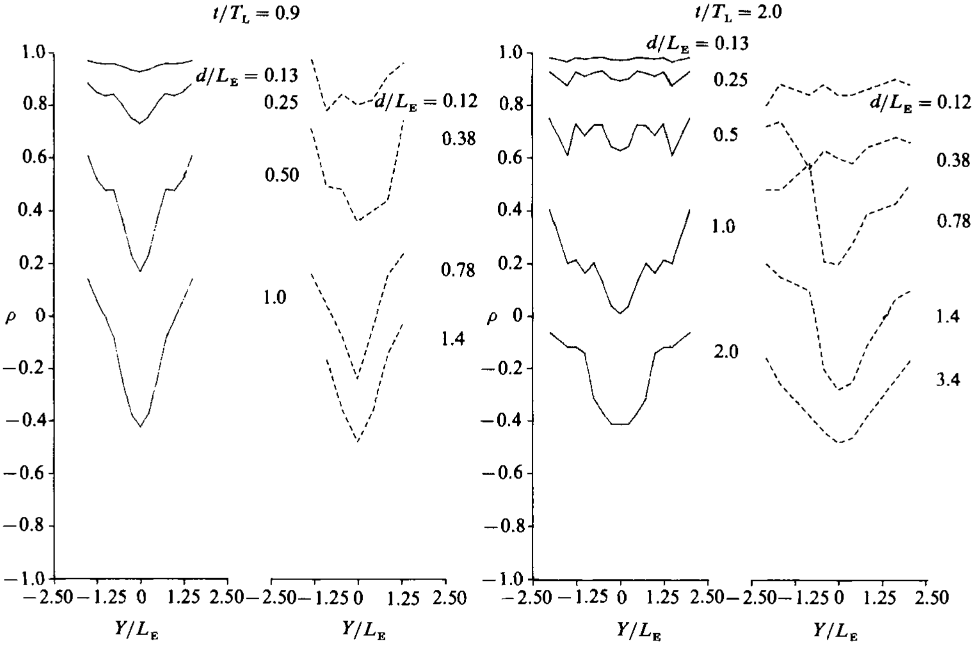


FIGURE 5. The dependence of the correlation ρ on the cross-wind direction, for different times and for different values of the spacing between the sources: ——, our model; ----, Warhaft's experimental values.

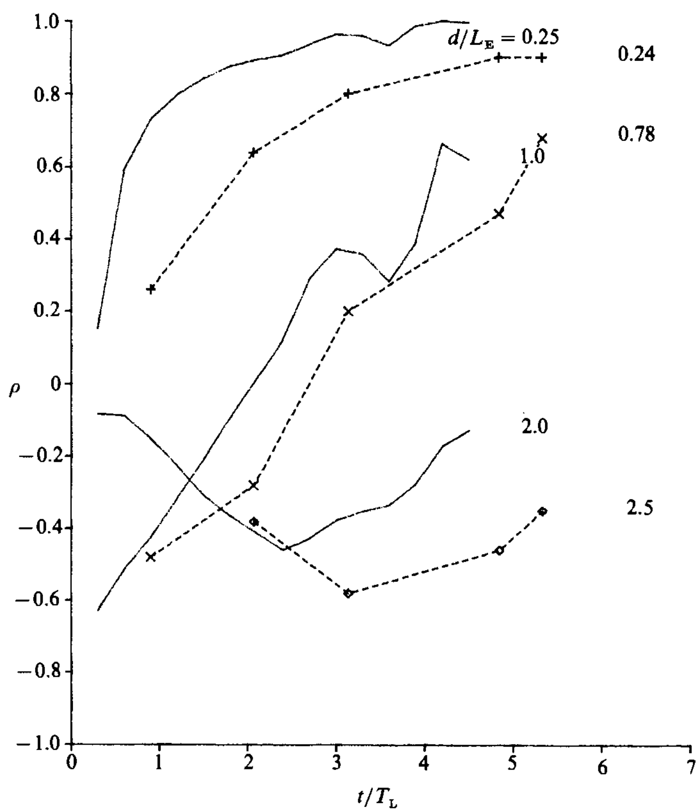


FIGURE 6. The evolution in time of ρ at the centreline between the two line sources. ——, our model; ----, Warhaft's experiments.

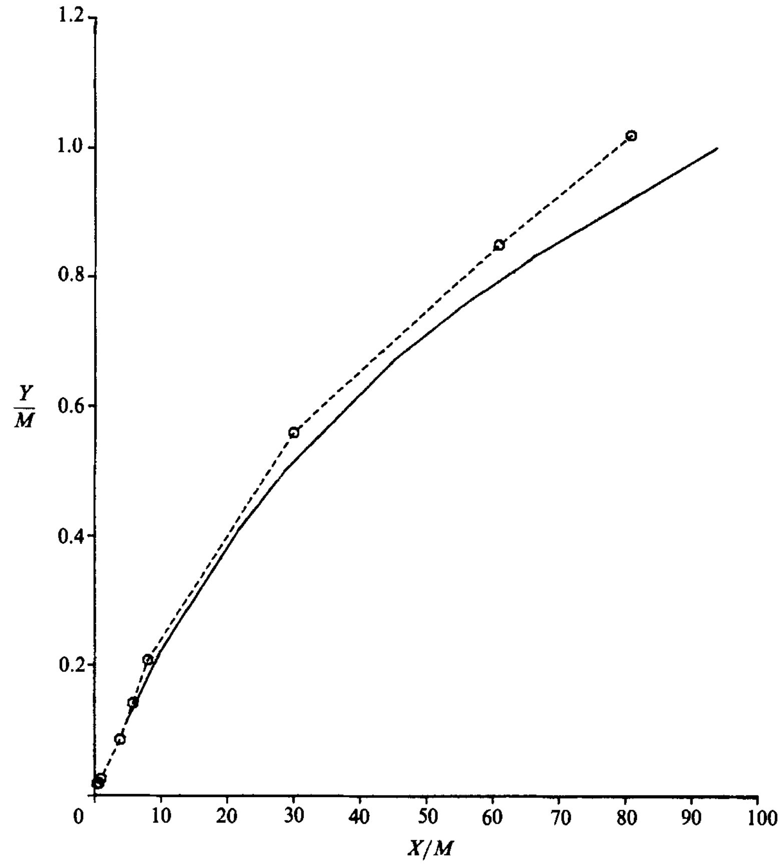


FIGURE 7. The dependence of the cloud width on the distance from the sources: ——, model prediction for stationary turbulence; ----, Warhaft's results. (Lengths are scaled by M, the grid parameter in Warhaft's experiments.)

intensity. Using this definition for T_L, we rescale Warhaft's measurements and compare them to our results, figure 6. We see that the experimental curves are in good agreement with the theoretical ones. We also compared Warhaft's results for the cloud width evolution with the theoretical prediction for the stationary turbulence (see Kaplan & Dinar 1988*b*; Taylor 1921). The cloud width $Y^{\frac{1}{2}}$ is defined as the distance from the cloud centre at which the concentration is half the maximum concentration:

$$\frac{Y^{\frac{1}{2}}}{L_E} = 1.18\frac{\sigma}{L_E} = \surd 2 \times 0.6\left(\exp\left(-t/T_L\right)+t/T_L-1\right)^{\frac{1}{2}}. \tag{4.2}$$

This comparison is presented in figure 7.

Another quantity that is compared to the experimental results is the model prediction of the fluctuations intensity s for a single source (formula (2.15)). The evolution in time of s is presented in figure 8 and one can see that the agreement is good. In the range for which the calculations were carried out, the fluctuation intensity decreases with time.

Figure 5 shows the cross-wind dependence of the experimental results versus the theoretical ones. (The experimental results are indicated by dashed lines.) Again, these results are in agreement with Warhaft's.

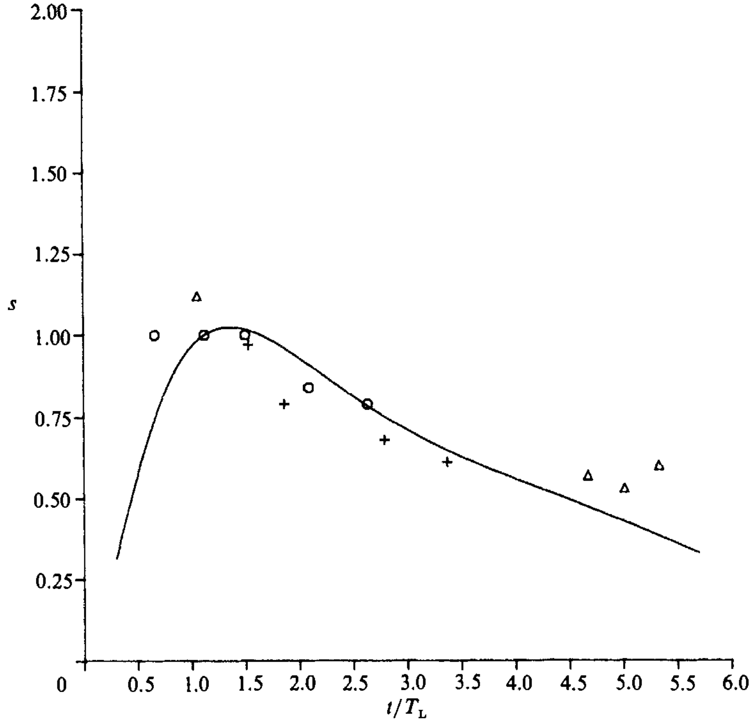


FIGURE 8. Time evolution of the fluctuation intensity for a single source. ——, model results; △, ○, +, Warhaft's experiments.

Less accurate results from field experiments are available in the work of Sawford *et al.* (1985). These experiments were carried out in inhomogeneous turbulence and no information is given on the turbulence scale. Qualitatively the correlation ρ evolution in time and dependence on source spacing is similar to that predicted by our model.

5. Summary

In this work, we have presented a method for calculating the cross-correlation of two passive scalars released in an isotropic homogeneous turbulence. The method was compared with experimental results and found to be in good agreement. It was found that the behaviour of the correlation depends on the Eulerian lengthscale and on the turbulence intensity. Results were presented in non-dimensional form, which enables us to use them with other values of the parameters. Extension of the model to include more than two sources is in progress.

Appendix

The stochastic process described by (2.1) and (2.2) is compatible with incompressible turbulent flow. The proof is carried out in two steps. First we shall prove (Lemma 1) that the stochastic process (2.1) and (2.2) describes an Eulerian distribution function of the velocity field which has a covariance matrix $K^{\alpha\beta}(|\boldsymbol{r}_i-\boldsymbol{r}_j|)$ which fulfills the condition

$$\sum_{\alpha}\frac{\partial}{\partial r_i^{\alpha}}K^{\alpha\beta}(\boldsymbol{r}_i,\boldsymbol{r}_j)=0 \quad \forall \boldsymbol{r}_i \neq \boldsymbol{r}_j. \tag{A 1}$$

(The summation convention is used in the appendix unless the 'Σ' sign is written explicitly.) Then we shall prove (Lemma 2) that a turbulent field is incompressible if and only if its Eulerian distribution function has a covariance $K^{\alpha\beta}(\boldsymbol{r}_i, \boldsymbol{r}_j)$ which satisfies (A 1).

LEMMA 1. *The covariance $K^{\alpha\beta}(\boldsymbol{r}_i, \boldsymbol{r}_j)$ of the Eulerian distribution function g_{E} described by the stochastic process (2.1) and (2.2) fulfills (A 1).*

Proof. It was proved by Thomson (1987) that the Eulerian distribution function of the field should be a solution of the equation

$$\frac{\partial g_{\mathrm{E}}}{\partial t} = -\frac{\partial}{\partial r_i^\alpha}(v_i^\alpha g_{\mathrm{E}}) + \frac{1}{T_{\mathrm{L}}}\frac{\partial}{\partial v_i^\alpha}(v_i^\alpha g_{\mathrm{E}}) + \frac{C^{\alpha\beta}(\boldsymbol{r}_i, \boldsymbol{r}_j)}{T_{\mathrm{L}}}\frac{\partial^2 g_{\mathrm{E}}}{\partial v_i^\alpha \partial v_j^\beta}. \tag{A 2}$$

The equation for the moment-generating function of g_{E} with respect to velocities which we denote by $\hat{g}_{\mathrm{E}}(\theta_1 \ldots \theta_N)$, is

$$\frac{\partial \hat{g}_{\mathrm{E}}}{\partial t} = -\frac{\partial}{\partial r_i^\alpha}\frac{\partial \hat{g}_{\mathrm{E}}}{\partial \theta_i^\alpha} - \frac{1}{T_{\mathrm{L}}}\theta_i^\alpha \frac{\partial \hat{g}_{\mathrm{E}}}{\partial \theta_i^\alpha} + \frac{C^{\alpha\beta}(\boldsymbol{r}_i, \boldsymbol{r}_j)}{T_{\mathrm{L}}}\theta_i^\alpha \theta_j^\beta \hat{g}_{\mathrm{E}}. \tag{A 3}$$

Substituting $\boldsymbol{\theta} = 0$ in (A 3) and using the fact that g_{E} is a distribution function and therefore its zero moment is 1, then

$$-\frac{\partial}{\partial r_i^\alpha}\frac{\partial \hat{g}_{\mathrm{E}}}{\partial \theta_i^\alpha} = 0.$$

This means that the divergence of the first moment of g_{E} is zero. On the other hand the stochastic process described by (2.1) and (2.2) using (2.3) and (2.5) is isotropic and therefore, the first moment of g_{E} must be proportional to $\boldsymbol{r}_i$ (see Batchelor 1956, p. 42). From this we can deduce that the first moment of g_{E} must be zero.

By taking derivatives of (A 3) with respect to θ_k^γ at the point $\theta = 0$ one gets

$$\frac{\partial}{\partial t}\frac{\partial \hat{g}_{\mathrm{E}}}{\partial \theta_k^\gamma} = -\frac{\partial}{\partial r_i^\alpha}\frac{\partial^2 \hat{g}_{\mathrm{E}}}{\partial \theta_i^\alpha \partial \theta_k^\gamma} - \frac{1}{T_{\mathrm{L}}}\frac{\partial \hat{g}_{\mathrm{E}}}{\partial \theta_k^\gamma}. \tag{A 4}$$

Since the first moments of g_{E} are zero, it follows from (A 4) that

$$\frac{\partial}{\partial r_i^\alpha}\frac{\partial^2 \hat{g}_{\mathrm{E}}}{\partial \theta_i^\alpha \partial \theta_k^\gamma} \equiv \frac{\partial}{\partial r_i^\alpha}K^{\alpha\beta}(\boldsymbol{r}_i, \boldsymbol{r}_k) = 0. \tag{A 5}$$

Therefore (A 1) holds and the Lemma is proved.

LEMMA 2(*a*). *If the Eulerian distribution function of velocities has a covariance $K^{\alpha\beta}(\boldsymbol{r}_i, \boldsymbol{r}_j)$ which fulfills the condition*

$$\sum_\alpha \frac{\partial}{\partial r_i^\alpha}K^{\alpha\beta}(\boldsymbol{r}_i, \boldsymbol{r}_j) = 0 \quad (\boldsymbol{r}_i \neq \boldsymbol{r}_j) \tag{A 6}$$

then the velocity field describes an incompressible flow.

Proof. Suppose the above claim is not correct, then there exists a realization of the flow which has a non-zero divergence at some point $\boldsymbol{r}_i$

$$\nabla \cdot \boldsymbol{v}^{(k)}(\boldsymbol{r}_i) \neq 0. \tag{A 7}$$

Then by multiplying the left hand side of (A 7) by $v^{(k)\beta}(\boldsymbol{r}_j)$ for $(\boldsymbol{r}_i \neq \boldsymbol{r}_j)$ and averaging over all realizations, we get

$$\frac{1}{M}\sum_{k=1}^{M}\sum_{\alpha}\frac{\partial}{\partial r_i^{\alpha}}v^{(k)\alpha}(\boldsymbol{r}_i)\,v^{(k)\beta}(\boldsymbol{r}_j)=\sum_{\alpha}\frac{\partial}{\partial r_i^{\alpha}}K^{\alpha\beta}(\boldsymbol{r}_i,\boldsymbol{r}_j)=0. \tag{A 8}$$

Taking the divergence at point $\boldsymbol{r}_j$, we have

$$\frac{1}{M}\sum_{k=1}^{M}\boldsymbol{\nabla}\cdot\boldsymbol{v}^{(k)}(\boldsymbol{r}_i)\,\boldsymbol{\nabla}\cdot\boldsymbol{v}^{(k)}(\boldsymbol{r}_j)=0. \tag{A 9}$$

Owing to the continuity of the velocity and its derivatives, it follows for $(\boldsymbol{r}_j\rightarrow\boldsymbol{r}_i)$

$$\frac{1}{M}\sum_{k=1}^{M}[\boldsymbol{\nabla}\cdot\boldsymbol{v}^{(k)}(\boldsymbol{r}_i)]^2=0 \tag{A 10}$$

which contradicts (A 7).

LEMMA 2(*b*). *If the flow is incompressible then* (A 6) *holds for incompressible flow*:

$$\boldsymbol{\nabla}\cdot\boldsymbol{v}^{(k)}(\boldsymbol{r}_i)=0 \tag{A 11}$$

at each point for every realization k of the field.

Multiplying (A 11) by $v^{\beta}(\boldsymbol{r}_j)$, $\boldsymbol{r}_j \neq \boldsymbol{r}_i$ and averaging over all realizations, we obtain

$$\langle\boldsymbol{\nabla}\cdot\boldsymbol{v}(\boldsymbol{r}_i)\,v^{\beta}(\boldsymbol{r}_j)\rangle=0. \tag{A 12}$$

Then by changing the order of averaging and differentiation we find

$$\sum_{\alpha}\frac{\partial}{\partial r_i^{\alpha}}K^{\alpha\beta}(\boldsymbol{r}_i,\boldsymbol{r}_j)=0. \tag{A 13}$$

REFERENCES

BATCHELOR, G. K. 1956 *The Theory of Homogeneous Turbulence*. Cambridge University Press.

DURBIN, P. A. 1980 A stochastic model of two particle dispersion and concentration fluctuations in homogeneous turbulence. *J. Fluid Mech.* **100**, 279–302.

DURBIN, P. A. 1982 Validity of the Lagrangian theory of concentration variance in turbulent dispersion. *Bull. Am. Phys. Soc.* **27**, 1180.

EGBERT, G. D. & BAKER, M. B. 1984 Comments on the effect of Gaussian particle-pair distribution function in statistical theory of concentration fluctuations in homogeneous turbulence. B. L. Sawford (1983) 109, 339–335; *Q. J. R. Met. Soc.* **110**, 1195–1199.

HANNA, S. R. 1979 Some statistics of Lagrangian and Eulerian wind fluctuation. *J. Appl. Met.* **18**, 518–525.

HANNA, S. R. 1981 Turbulent energy and Lagrangian time scales in the planetary boundary layer. In *5th Symposium on Turbulence, Diffusion, and Air Pollution*, pp. 61–62. AMS.

KAPLAN, H. & DINAR, N. 1988*a* A stochastic model for dispersion and concentration distribution in homogeneous turbulence. *J. Fluid Mech.* **190**, 121–140.

KAPLAN, H. & DINAR, N. 1988*b* A three-dimensional stochastic model for concentration fluctuation statistics in isotropic homogeneous turbulence. *J. Comput. Phys.* **79**, 317–335.

SAWFORD, B. L. 1983 The effect of Gaussian particle pair distribution function in the statistical theory of concentration fluctuations in homogeneous turbulence. *Q. J. R. Met. Soc.* **109**, 339–354.

SAWFORD, B. L., FROST, C. C. & ALLAN, T. C. 1985 Atmospheric boundary-layer measurements of concentration statistics from isolated and multiple sources. *Boundary-Layer Met.* **31**, 249–268.

SAWFORD, B. L. & HUNT, J. C. R. 1986 Effect of turbulence structure molecular diffusion and source size on fluctuations of concentration in homogeneous turbulence. *J. Fluid Mech.* **165**, 374–400.

TAYLOR, G. I. 1921 Diffusion by continuous movements. *Proc. Lond. Math. Soc.* (2), **20**, 196–211.

THOMSON, D. J. 1986 A random walk model of dispersion in turbulent flow and its application to dispersion in a valley. *Q. J. R. Met. Soc.* **112**, 511–530.

THOMSON, D. J. 1987 Criteria for the selection of stochastic models of particles trajectories in turbulent flows. *J. Fluid Mech.* **180**, 529–556.

WARHAFT, Z. 1984 The interference of thermal fields for line sources in grid turbulence. *J. Fluid Mech.* **144**, 363–381.